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On a new method for constructing exact solutions of the nonlinear differential equations of mathematical physics

A F Barannyk and I I Yuryk

Department of Applied Research, Institute of Mathematics of the National Academy of Sciences of Ukraine, 3 Tereshchenkivska Str., Kyiv 4, 252 004 Ukraine

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Abstract. A simple new method for constructing solutions of multidimensional nonlinear wave equations is proposed.

1. Introduction

The method of symmetry reduction of an equation to equations with fewer variables, in particular, to ordinary differential equations [1–3] is among the most efficient methods for constructing solutions of nonlinear equations in mathematical physics. This method is based on investigation of the subgroup structure of an invariance group of a given differential equation. Solutions being obtained in this way are invariant with respect to a subgroup of the invariance group of the equation. It is worth noting that the invariance imposes very severe constraints on solutions. For this reason, the symmetry reduction does not allow one to obtain a sufficiently wide classes of solutions in many cases.

The idea of the conditional invariance of differential equations, proposed in [3–6], is particularly interesting. By conditional symmetry of an equation, one means the symmetry of some solution set. For a lot of the important nonlinear equations of mathematical physics, there exist solution subsets, the symmetry of which is essentially different from that of the whole solution set. One chooses such solution subsets, as a rule, with the help of additional conditions representing partial differential equations. The description of these additional conditions in the explicit form is a difficult problem and unfortunately there are no efficient methods to solve it.

In this paper, we propose a simple method for constructing some classes of exact solutions to the nonlinear equations of mathematical physics. We notice that the idea of this method was formulated by Fushchych and Barannyk [7]. The essence of the method is the following. Let we have a partial differential equation

$$F\big(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}\big) = 0 \tag{1}$$

where $u = u(x)$, $x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}_{1,n}$, $\underset{m}{u}$ is a collection of all possible derivatives of order m, and let equation (1) have a nontrivial symmetry algebra. To construct solutions of equation (1), we use the symmetry (or conditional symmetry) ansatz [3]. Suppose that it is of the form

$$u = f(x)\,\varphi(\omega_1, \ldots, \omega_k) + g(x) \tag{2}$$

where $\omega_1 = \omega_1(x_0, x_1, \dots, x_n), \dots, \omega_k = \omega_k(x_0, x_1, \dots, x_n)$ are new independent variables. Ansatz (2) singles out some subset S from the whole solution set of equation (1). Construct (if it is possible) a new ansatz

$$u = f(x)\,\varphi(\omega_1, \dots, \omega_k, \omega_{k+1}, \dots, \omega_l) + g(x) \tag{3}$$

which is a generalization of ansatz (2). Here $\omega_{k+1}, \dots, \omega_l$ are new variables that should be determined. We choose the variables $\omega_{k+1}, \dots, \omega_l$ from the condition that the reduced equation corresponding to ansatz (3) coincides with the reduced equation corresponding to ansatz (2). Ansatz (3) singles out a subset S_1 of solutions to equation (1), being an extension of the subset S. If solutions of the subset S are known, then one can also construct solutions of the subset S_1. These solutions are constructed in the following way. Let $u = u(x, C_1, \dots, C_t)$ be a multiparameter solution set of the form (2) of equation (1), where $C_1, \dots, C_t$ are arbitrary constants. We shall obtain a more general solution set of equation (1) if we take constants C_i in the solution $u = u(x, C_1, \dots, C_t)$ to be arbitrary smooth functions of $\omega_{k+1}, \dots, \omega_l$.

Basic aspects of our approach are presented by the examples of d'Alembert, Liouville and eikonal equations.

2. Nonlinear d'Alembert equations

Let us consider a nonlinear Poincaré-invariant d'Alembert equation

$$\Box u + F(u) = 0 \tag{4}$$

where

$$\Box u = \frac{\partial^2 u}{\partial x_0^2} - \frac{\partial^2 u}{\partial x_1^2} - \cdots - \frac{\partial^2 u}{\partial x_n^2}$$

and $F(u)$ is an arbitrary smooth function. References [3, 8–10] are devoted to the construction of exact solutions to equation (4) for different restrictions on the function $F(x)$. The majority of these solutions is invariant with respect to a subgroup of the invariance group of equation (4), i.e. they are Lie solutions. One of the methods for constructing solutions is the method of symmetry reduction of equation (4) to ordinary differential equations. The essence of this method for equation (4) consists in the following.

Equation (4) is invariant under the Poincaré algebra $AP(1, n)$ with the basis elements

$$J_{0a} = x_0\partial_a + x_a\partial_0 \qquad J_{ab} = x_b\partial_a - x_a\partial_b$$
$$P_0 = \partial_0 \qquad P_a = \partial_a \qquad (a, b = 1, 2, \dots, n).$$

Let L be an arbitrary rank n subalgebra of the algebra $AP(1, n)$. The subalgebra L has two main invariants u, $\omega = \omega(x_0, x_1, \dots, x_n)$. The ansatz $u = \varphi(\omega)$ corresponding to the subalgebra L reduces equation (4) to the ordinary differential equation

$$\ddot{\varphi}(\nabla\omega)^2 + \dot{\varphi}\Box\omega + F(\varphi) = 0 \tag{5}$$

where

$$(\nabla\omega)^2 \equiv \left(\frac{\partial\omega}{\partial x_0}\right)^2 - \left(\frac{\partial\omega}{\partial x_1}\right)^2 - \cdots - \left(\frac{\partial\omega}{\partial x_n}\right)^2.$$

Such a reduction is called the *symmetry reduction*, and the ansatz is called the *symmetry ansatz*. There exist eight types of nonequivalent rank n subalgebras of the algebra $AP(1, n)$ [8]. In table 1, we write out these subalgebras, their invariants and values of $(\nabla\omega)^2$, $\Box\omega$ for each invariant.

Table 1.

N	Algebra	Invariant ω	$(\nabla\omega)^2$	$\Box\omega$
1	$P_1, \ldots, P_n$	x_0	1	0
2	$P_0, P_1, \ldots, P_{n-1}$	x_n	-1	0
3	$P_1, \ldots, P_{n-1}, J_{0n}$	$\left(x_0^2 - x_n^2\right)^2$	1	$\dfrac{1}{\omega}$
4	$J_{ab} \quad (a, b = 1, \ldots, k),$ $P_{k+1}, \ldots, P_n, P_0 \quad (k \geqslant 2)$	$\left(x_1^2 + \cdots + x_k^2\right)^{1/2}$	-1	$-\dfrac{k-1}{\omega}$
5	$G_a = J_{0a} - J_{ak}, J_{ab}$ $(a, b = 1, \ldots, k-1)$ $J_{0k}, P_{k+1}, \ldots, P_n \quad (k \geqslant 1)$	$\left(x_0^2 - x_1^2 - \cdots - x_k^2\right)^{1/2}$	1	$\dfrac{k}{\omega}$
6	$P_1, \ldots, P_{n-2}, P_0 + P_n,$ $J_{0n} + \alpha P_{n-1}$	$\alpha \ln(x_0 - x_n) + x_{n-1}$	-1	0
7	$P_0 + P_n, P_1, \ldots, P_{n-1}$	$x_0 - x_n$	0	0
8	$P_a \quad (a = 1, \ldots, n-2),$ $G_{n-1} + P_0 - P_n, P_0 + P_n$	$(x_0 - x_n)^2 - 4x_{n-1}$	-1	0

The method proposed in [12] of reduction of equation (4) to ODEs is a generalization of the symmetry reduction method. Equation (4) is reduced to ODEs with the help of the ansatz $u = \varphi(\omega)$, where $\omega = \omega(x)$ is a new variable, if $\omega(x)$ satisfies the equations

$$\Box\omega = F_1(\omega) \qquad (\nabla\omega)^2 = F_2(\omega). \tag{6}$$

Here F_1, F_2 are arbitrary smooth functions depending only on ω.

Thus, if we construct all solutions to system (6), we get the set of all values of the variable ω, for which the ansatz $u = \varphi(\omega)$ reduces equation (4) to ODEs in the variable ω. References [11, 12] are devoted to the investigation of system (6).

Note, however, that ansätze obtained by solving system (6), do not exhaust the set of all ansätze reducing equation (4) to ordinary differential equations. For this purpose, let us consider the process of finding generalized ansätze (3) on the known symmetry ansätze (2) of equation (4).

(i) Consider the symmetry ansatz $u = \varphi(\omega_1)$ for equation (4), where $\omega_1 = \left(x_0^2 - x_1^2 - \cdots - x_k^2\right)$, $k \geqslant 2$. The ansatz reduces equation (4) to the equation

$$\varphi_{11} + \frac{k}{\omega_1}\varphi_1 + F(\omega_1) = 0 \tag{7}$$

where $\varphi_{11} = \mathrm{d}^2\varphi/\mathrm{d}\omega_1^2$, $\varphi_1 = \mathrm{d}\varphi/\mathrm{d}\omega_1$. This ansatz should be regarded as a partial case of the more general ansatz $u = \varphi(\omega_1, \omega_2)$, where ω_2 is an unknown variable. The ansatz $u = \varphi(\omega_1, \omega_2)$ reduces equation (4) to the equation

$$\varphi_{11} + \frac{k}{\omega_1}\varphi_1 + 2\varphi_{12}(\nabla\omega_1 \cdot \nabla\omega_2) + \varphi_2\Box\omega_2 + \varphi_{22}(\nabla\omega_2)^2 + F(\varphi) = 0 \tag{8}$$

where

$$\nabla\omega_1 \cdot \nabla\omega_2 = \frac{\partial\omega_1}{\partial x_0} \cdot \frac{\partial\omega_2}{\partial x_0} - \frac{\partial\omega_1}{\partial x_1} \cdot \frac{\partial\omega_2}{\partial x_1} - \cdots - \frac{\partial\omega_1}{\partial x_n} \cdot \frac{\partial\omega_2}{\partial x_n}.$$

Let us impose the condition on equation (8), under which equation (8) coincides with the reduced equation (7). Under such an assumption, equation (8) decomposes into two equations

$$\varphi_{11} + \frac{k}{\omega_1}\varphi_1 + F(\varphi) = 0 \tag{9}$$

$$2\varphi_{12}(\nabla\omega_1 \cdot \nabla\omega_2) + \varphi_{22}(\nabla\omega_2)^2 + \varphi_{12}\Box\omega_2 = 0. \tag{10}$$

Equation (10) will be fulfilled for an arbitrary function φ if we impose the conditions

$$\Box\omega_2 = 0, \qquad (\nabla\omega_2)^2 = 0 \tag{11}$$

$$\nabla\omega_1 \cdot \nabla\omega_2 = 0 \tag{12}$$

on the variable ω_2. Therefore, if we choose the variable ω_2 such that conditions (11) and (12) are satisfied, then the multidimensional equation (4) is reduced to the ordinary differential equation (7) and solutions of the latter equation give us solutions of equation (4). So, the problem of reduction is reduced to the construction of general or partial solutions to the system (11) and (12).

The overdetermined system (11) is studied in detail in [13, 14], where a wide class of solutions to system (11) is constructed. These solutions are constructed in the following way. Let us consider a linear algebraic equation in variables $x_0, x_1, \dots, x_n$ with coefficients depending on the unknown ω_2:

$$a_0(\omega_2)x_0 - a_1(\omega_2)x_1 - \cdots - a_n(\omega_2)x_n - b(\omega_2) = 0. \tag{13}$$

Let the coefficients of this equation represent analytic functions of ω_2 satisfying the condition

$$[a_0(\omega_2)]^2 - [a_1(\omega_2)]^2 - \cdots - [a_n(\omega_2)]^2 = 0.$$

Suppose that equation (13) is solvable for ω_2 and let a solution of this equation represent some real or complex function

$$\omega_2(x_0, x_1, \dots, x_n). \tag{14}$$

Then function (14) is a solution to system (11). Single out those solutions (14), that possess the additional property $\nabla\omega_1 \cdot \nabla\omega_2 = 0$. It is obvious that

$$\frac{\partial\omega_2}{\partial x_0} = -\frac{a_0}{\delta'}, \quad \frac{\partial\omega_2}{\partial x_1} = \frac{a_1}{\delta'}, \quad \dots, \quad \frac{\partial\omega_2}{\partial x_n} = \frac{a_n}{\delta'}$$

where

$$\delta(\omega_2) \equiv a_0(\omega_2)x_0 - a_1(\omega_2)x_1 - \cdots - a_n(\omega_2)x_n - b(\omega_2)$$

and δ' is the derivative of δ with respect to ω_2. Since

$$\frac{\partial\omega_1}{\partial x_0} = \frac{x_0}{\omega_1}, \quad \frac{\partial\omega_1}{\partial x_1} = -\frac{x_1}{\omega_1}, \quad \dots, \quad \frac{\partial\omega_1}{\partial x_n} = -\frac{x_n}{\omega_1}$$

we have

$$\nabla\omega_1 \cdot \nabla\omega_2 = -\frac{1}{\omega_1\delta'}(a_0x_0 - a_1x_1 - \cdots - a_nx_n).$$

Hence, with regard for (13), the equality $\nabla\omega_1 \cdot \nabla\omega_2 = 0$ is fulfilled if and only if $b(\omega_2) = 0$. Therefore, we have constructed the wide class of ansätze reducing the d'Alembert equation to ordinary differential equations. The arbitrariness in choosing the function ω_2 may be used to satisfy some additional conditions (initial, boundary and so on).

(ii) The symmetry ansatz $u = \varphi(\omega_1)$, $\omega_1 = \left(x_1^2 + \cdots + x_l^2\right)^{1/2}$, $1 \leqslant l < n-1$, is generalized in the following way. Let ω_2 be an arbitrary solution to the system of equations

$$\begin{gathered}\frac{\partial^2\omega}{\partial x_0^2} - \frac{\partial^2\omega}{\partial x_{l+1}^2} - \cdots - \frac{\partial^2\omega}{\partial x_n^2} = 0 \\ \left(\frac{\partial\omega}{\partial x_0}\right)^2 - \left(\frac{\partial\omega}{\partial x_{l+1}}\right)^2 - \cdots - \left(\frac{\partial\omega}{\partial x_n}\right)^2 = 0.\end{gathered} \tag{15}$$

The ansatz $u = \varphi(\omega_1, \omega_2)$ reduces equation (4) to the equation

$$-\frac{\mathrm{d}^2\varphi}{\mathrm{d}\omega_1^2} - \frac{k-1}{\omega_1}\frac{\mathrm{d}\varphi}{\mathrm{d}\omega_1} + F(\varphi) = 0.$$

If $l = n-1$, then the ansatz $u = \varphi(\omega_1, \omega_2)$, $\omega_2 = x_0 - x_n$ is a generalization of the symmetry ansatz $u = \varphi(\omega_1)$.

Ansätze corresponding to subalgebras 2, 6 and 8 in table 1, are particular cases of the ansatz constructed above. In a similar way, one can obtain wide classes of ansätze reducing equation (4) to two-, three-dimensional and so on equations. Let us present some of them.

(iii) The ansatz $u = \varphi(\omega_1, \ldots, \omega_l, \omega_{l+1})$, where $\omega_1 = x_1, \ldots, \omega_l = x_l$, ω_{l+1} is an arbitrary solution of system (15), $l \leqslant n-1$, is a generalization of the symmetry ansatz $u = \varphi(\omega_1, \ldots, \omega_l)$ and reduces equation (4) to the equation

$$-\frac{\partial^2\varphi}{\partial\omega_1^2} - \frac{\partial^2\varphi}{\partial\omega_2^2} - \cdots - \frac{\partial^2\varphi}{\partial\omega_l^2} + F(\varphi) = 0.$$

(iv) The ansatz $u = \varphi(\omega_1, \ldots, \omega_s, \omega_{s+1})$, where $\omega_1 = \left(x_0^2 - x_1^2 - \cdots - x_l^2\right)^{1/2}$, $\omega_2 = x_{l+1}, \ldots, \omega_s = x_{l+s-1}$, $l \geqslant 2$, $l + s - 1 \leqslant n$, ω_{s+1} is an arbitrary solution of the system

$$\Box\omega_{s+1} = 0 \qquad (\nabla\omega_{s+1})^2 = 0 \qquad \nabla\omega_i \cdot \nabla\omega_{s+1} = 0 \qquad i = 1, 2, \ldots, s \tag{16}$$

is a generalization of the symmetry ansatz $u = \varphi(\omega_1, \ldots, \omega_s)$ and reduces equation (4) to the equation

$$\varphi_{11} - \frac{l}{\omega_1}\varphi_1 - \varphi_{22} - \cdots - \varphi_{ss} + F(\varphi) = 0.$$

Let us construct in the way described above some classes of exact solutions of the equation

$$\Box u + \lambda u^k = 0 \qquad k \neq 1. \tag{17}$$

The following solution of equation (17) is obtained in [10]:

$$u^{1-k} = \sigma(k, l)\left(x_1^2 + \cdots + x_l^2\right) \tag{18}$$

where

$$\sigma(k, l) = \frac{\lambda(1-k)^2}{2(l - lk + 2k)} \qquad l = 1, 2, \ldots, n.$$

Solution (18) defines a multiparameter solution set

$$u^{1-k} = \sigma(k, l)\left[(x_1 + C_1)^2 + \cdots + (x_l + C_l)^2\right]$$

where $C_1, \ldots, C_l$ are arbitrary constants. Hence, according to (iii), we obtain the following set of solutions to equation (17) for $l \leqslant n-1$:

$$u^{1-k} = \sigma(k, l)\left[(x_1 + h_1(\omega))^2 + \cdots + (x_l + h_l(\omega))^2\right] \qquad k \neq \frac{l}{l-2}$$

where ω is an arbitrary solution of system (15) and $h_1(\omega), \dots, h_l(\omega)$ are arbitrary twice differentiable functions of ω. In particular, if $n = 3$ and $l = 1$, then equation (17) possesses in the space $\mathbb{R}_{1,3}$ the solution set

$$u^{1-k} = \frac{\lambda(1-k)^2}{2(1+k)}[x_1 + h_1(\omega)]^2 \qquad k \neq -1.$$

Next, let us consider the following solution of equation (4) [10]:

$$u^{1-k} = \sigma(k, s)\left(x_0^2 - x_1^2 - \cdots - x_s^2\right) \qquad s = 2, \dots, n \tag{19}$$

where

$$\sigma(k, s) = -\frac{\lambda(1-k)^2}{2(s - ks + k + 1)} \qquad k \neq \frac{s+1}{s-1}.$$

Solution (19) defines the multiparameter solution set

$$u^{1-k} = \sigma(k, s)\left[x_0^2 - x_1^2 - \cdots - x_l^2 - (x_{l+1} + C_{l+1})^2 - \cdots - (x_s + C_s)^2\right]$$

where $C_{l+1}, \dots, C_s$ are arbitrary constants. According to (iv) we obtain the following solution set for $l \geqslant 2$:

$$u^{1-k} = \sigma(k, s)\left[x_0^2 - x_1^2 - \cdots - x_l^2 - (x_{l+1} + h_{l+1}(\omega))^2 - \cdots - (x_s + h_s(\omega))^2\right]$$

where ω is an arbitrary solution of system (16), and $h_{l+1}(\omega), \dots, h_s(\omega)$ are arbitrary twice differentiable functions. In particular, if $l = 2$ and $s = 3$, then equation (4) possesses in the space $\mathbb{R}_{1,3}$ the following solution set:

$$u^{1-k} = \frac{\lambda(1-k)^2}{4(k-2)}\left[x_0^2 - x_1^2 - x_2^2 - (x_3 - h_3(\omega))^2\right] \qquad k \neq 2.$$

The equation

$$\Box u + 6u^2 = 0 \tag{20}$$

possesses the solution $u = \mathcal{P}(x_3 + C_2)$, where $\mathcal{P}(x_3 + C_2)$ is an elliptic Weierstrass function with the invariants $g_2 = 0$ and $g_3 = C_1$. Therefore, according to (iii) we get the following set of solutions of equation (20):

$$u = \mathcal{P}(x_3 + h(\omega))$$

where ω is an arbitrary solution to system (15) and $h(\omega)$ is an arbitrary twice differentiable function of ω.

Next consider the Liouville equation

$$\Box u + \lambda \exp u = 0. \tag{21}$$

The symmetry ansatz $u = \varphi(\omega_1)$, $\omega_1 = x_3$, reduces equation (21) to the equation

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}\omega_1^2} = \lambda \exp \varphi(\omega_1).$$

Integrating this equation, we obtain that φ coincides with one of the following functions:

$$\ln\left\{\left(-\frac{C_1}{2\lambda}\sec^2\left[\frac{\sqrt{-C_1}}{2}(\omega_1 + C_2)\right]\right)\right\} \qquad (C_1 < 0,\ \lambda > 0,\ C_2 \in \mathbb{R})$$

$$\ln\left\{\frac{2C_1C_2\exp\left(\sqrt{C_1}\omega_1\right)}{\lambda\left[1 - C_2\exp\left(\sqrt{C_1}\omega_1\right)\right]^2}\right\} \qquad (C_1 > 0,\ \lambda C_2 > 0)$$

$$-\ln\left(\sqrt{\frac{\lambda}{2}}\omega_1 + C\right)^2.$$

Hence, according to (iii) we get the following solution set for equation (21):

$$u = \ln\left\{\left(-\frac{h_1(\omega)}{2\lambda}\sec^2\left[\frac{\sqrt{-h_1(\omega)}}{2}(\omega_1 + h_2(\omega))\right]\right)\right\} \qquad (h_1(\omega) < 0,\ \lambda > 0)$$

$$u = \ln\left\{\frac{2h_1(\omega)h_2(\omega)\exp\left(\sqrt{h_1(\omega)}\omega_1\right)}{\lambda\left[1 - h_2(\omega)\exp\left(\sqrt{h_1(\omega)}\omega_1\right)\right]^2}\right\} \qquad (h_1(\omega) > 0,\ \lambda h_2(\omega) > 0)$$

$$u = -\ln\left(\sqrt{\frac{\lambda}{2}}\omega_1 + h(\omega)\right)^2$$

where $h_1(\omega), h_2(\omega), h(\omega)$ are arbitrary twice differentiable functions; ω is an arbitrary solution to system (15).

Using, for example, the solution to the Liouville equation (21) [10]

$$u = \ln\frac{2(s-2)}{\lambda\left[x_0^2 - x_1^2 - \cdots - x_s^2\right]} \qquad s \neq 2$$

we obtain the wide class of solutions to the Liouville equation

$$u = \ln\frac{2(s-2)}{\lambda\left[x_0^2 - x_1^2 - \cdots - x_l^2 - (x_{l+1} + h_{l+1}(\omega))^2 - \cdots - (x_s + h_s(\omega))^2\right]}$$

where ω is an arbitrary solution to system (16), and $h_{l+1}(\omega), \ldots, h_s(\omega)$ are arbitrary twice differentiable functions. If $s = 3$, then equation (21) possesses in the space $\mathbb{R}_{1,3}$ the following solution set:

$$u = \ln\frac{2}{\lambda\left[x_0^2 - x_1^2 - x_2^2 - (x_3 + h_3(\omega))^2\right]}.$$

Let us consider now the sine–Gordon equation

$$\Box u + \sin u = 0.$$

In an analogous way, we get the following solutions:

$$u = 4\arctan h_1(\omega)\,\mathrm{e}^{\varepsilon_0 x_3} - \tfrac{1}{2}(1-\varepsilon)\pi \qquad \varepsilon_0 = \pm 1 \quad \varepsilon = \pm 1$$

$$u = 2\arccos[\mathrm{dn}(x_3 + h_1(\omega)), m] + \tfrac{1}{2}(1+\varepsilon)\pi \qquad 0 < m < 1$$

$$u = 2\arccos\left[\mathrm{cn}\left(\frac{x_3 + h_1(\omega)}{m}\right), m\right] + \tfrac{1}{2}(1+\varepsilon)\pi \qquad 0 < m < 1$$

where $h_1(\omega)$ is an arbitrary twice differentiable function, ω is an arbitrary solution to system (15).

3. Eikonal equation

Consider the eikonal equation

$$\left(\frac{\partial u}{\partial x_0}\right)^2 - \left(\frac{\partial u}{\partial x_1}\right)^2 - \left(\frac{\partial u}{\partial x_2}\right)^2 - \left(\frac{\partial u}{\partial x_3}\right)^2 = 1. \qquad (22)$$

The symmetry ansatz $u = \varphi(\omega_1)$, $\omega_1 = x_0^2 - x_1^2 - x_2^2 - x_3^2$, reduces equation (22) to the equation

$$4\omega_1\left(\frac{\partial\varphi}{\partial\omega_1}\right)^2 - 1 = 0. \qquad (23)$$

We shall look for a generalized ansatz in the form $u = \varphi(\omega_1, \omega_2)$. This ansatz reduces equation (22) to the equation

$$4\omega_1\left(\frac{\partial\varphi}{\partial\omega_1}\right)^2 + 2(\nabla\omega_1 \cdot \nabla\omega_2)\frac{\partial\varphi}{\partial\omega_1} + (\nabla\omega_2)^2\left(\frac{\partial\varphi}{\partial\omega_2}\right)^2 = 1. \tag{24}$$

Impose the condition on equation (24), under which equation (24) coincides with equation (23). It is obvious that this condition will be fulfilled if we impose the conditions

$$(\nabla\omega_2)^2 = 0 \qquad \nabla\omega_1 \cdot \nabla\omega_2 = 0 \tag{25}$$

on the variable ω_2. Having solved system (25), we get the explicit form of the variable ω_2. It is easy to see that an arbitrary function of a solution to system (25) is also a solution to this system.

Having integrated equation (23), we obtain $(u + C)^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$, where C is an arbitrary constant. We shall obtain a more general solution set for the eikonal equation if we take C to be an arbitrary solution to system (25).

The symmetry ansatz $u = \varphi(\omega_1, \omega_2)$, $\omega_1 = x_0^2 - x_1^2 - x_2^2$, $\omega_2 = x_3$ can be generalized in the following way. Let ω_3 be an arbitrary solution to the system of equations

$$\begin{aligned} &\left(\frac{\partial\omega_3}{\partial x_0}\right)^2 - \left(\frac{\partial\omega_3}{\partial x_1}\right)^2 - \left(\frac{\partial\omega_3}{\partial x_2}\right)^2 = 0 \\ &x_0\frac{\partial\omega_3}{\partial x_0} + x_1\frac{\partial\omega_3}{\partial x_1} + x_3\frac{\partial\omega_3}{\partial x_2} = 0. \end{aligned} \tag{26}$$

Then the ansatz $u = \varphi(\omega_1, \omega_2, \omega_3)$ reduces the eikonal equation to the equation

$$4\omega_1\left(\frac{\partial\varphi}{\partial\omega_1}\right)^2 - \left(\frac{\partial\varphi}{\partial\omega_2}\right)^2 - 1 = 0. \tag{27}$$

Equation (27) possesses the solution [10]

$$\begin{aligned} &\varphi = \frac{C_1^2 + 1}{2C_1}\left(x_0^2 - x_1^2 - x_2^2\right)^{1/2} + \frac{C_1^2 - 1}{2C_1}x_3 + C_2 \\ &(\varphi + C_2)^2 = x_0^2 - x_1^2 - x_2^2 - (x_3 + C_1)^2 \end{aligned}$$

that can be easily found by using the symmetry reduction method of equation (27) to an ordinary differential equation. Having replaced arbitrary constants C_1 and C_2 by arbitrary functions $h_1(\omega)$ and $h_2(\omega)$, we get the more wide classes of exact solutions to the eikonal equation:

$$\begin{aligned} &u = \frac{h_1(\omega_3)^2 + 1}{2h_1(\omega_3)}\left(x_0^2 - x_1^2 - x_2^2\right)^{1/2} + \frac{h_1(\omega_3)^2 - 1}{2h_1(\omega_3)}x_3 + h_2(\omega_3) \\ &(u + h_2(\omega_3))^2 = x_0^2 - x_1^2 - x_2^2 - (x_3 + h_1(\omega_3))^2. \end{aligned}$$

Let us note, since the Born–Infeld equation is a differential consequence of the eikonal equation [3], we also constructed wide classes of exact solutions of the Born–Infeld equation.

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